

# On the solutions of the Yang-Baxter equations with general inhomogeneous eight-vertex $R$ -matrix: Relations with Zamolodchikov's tetrahedral algebra

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## Abstract

We present most general one-parametric solutions of the Yang-Baxter equations (YBE) for one spectral parameter dependent  $R_{ij}(u)$ -matrices of the six- and eight-vertex models, where the only constraint is the particle number conservation by mod(2). A complete classification of the solutions is performed. We have obtained also two spectral parameter dependent particular solutions  $R_{ij}(u, v)$  of YBE. The application of the non-homogeneous solutions to construction of Zamolodchikov's tetrahedral algebra is discussed.

## 1 Introduction

Yang-Baxter equations (YBE) play an important role in theory of two-dimensional integrable models [1, 2, 3, 4, 5, 6, 7]. They ensure the presence of infinite amount of conservation laws making model integrable. Integrable models arise in many areas of physics (statistical mechanics, high energy physics, condensed matter physics, string theory, atomic and molecular physics), as well as the mathematical methods and tools developed in the theory of integrable systems have broad applications in different branches of modern physics (e.g. see the works [3, 4, 5, 8, 9, 10, 11] and references therein). A major

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element in the theory of two-dimensional integrable models is  $R$ -matrix, which should satisfy YBE.  $R$ -matrices are classified by degrees of freedom of the chain sites, where they are acting, and by the symmetries of the model. Usually minimal, simplest solutions of YBE are enough to characterize the model. The  $R$ -matrices of the anisotropic Heisenberg, Hubbard, Uimin-Lai-Sutherland models are just simplest solutions of corresponding YBE and they carry most important properties of the corresponding classes. However, one can expect, that there can be a generalization of the model by keeping integrability, which, nevertheless, will correspond to another physical situation. In this context a substantial question is rising about the properties of the statistical (Boltzmann) weights in two dimensional statistical physics [4, 5, 7], or the scattering  $R$ -matrices in the  $1 + 1$  quantum field theories [1, 6], which maintain the integrability. The general form of the spectral parameter dependent YBE is the following

$$R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v). \quad (1.1)$$

In this paper we consider the one parametric class of  $R$ -matrices,  $R(u, w) = R(u - w)$ .  $R_{12}$  matrix acts on the direct product of two states,  $|i_1\rangle|i_2\rangle$ . The variables  $i_k$  describe the degrees of states at the lattice sites (in context of 1-dimensional quantum chain models or 2-dimensional models in classical statistical mechanics) or the states of the scattering particles (in context of  $(1 + 1)$ -dimensional quantum scattering theory). Let us constrain ourselves to the case of two dimensional quantum spaces at the sites, namely when  $i_k$  can have only two values (spin- $\frac{1}{2}$  system).

Usually some characteristics of the  $R$ -matrix of the model are fixed on the basis of symmetry properties of the underlying physical problem. Here we fix only the structure of  $R$ -matrices, i.e the non-vanishing elements positions. The only symmetries, which is taken into account, is the "particle number" conservation by mod(2) ( $\mathcal{Z}_2$  grading symmetry). In matrix representation it means

$$R_{ij}^{kr} \neq 0 \quad \text{if} \quad i + j + k + r = 0(\text{mod } 2). \quad (1.2)$$

If  $i, j, \dots$  take only two values 0, 1, then  $R$  has a following  $4 \times 4$  matrix form

$$R(u-v) = \begin{pmatrix} R_{00}^{00} & 0 & 0 & R_{00}^{11} \\ 0 & R_{01}^{01} & R_{01}^{10} & 0 \\ 0 & R_{10}^{01} & R_{10}^{10} & 0 \\ R_{11}^{00} & 0 & 0 & R_{11}^{11} \end{pmatrix}, \quad (1.3)$$

corresponding to the  $R$ -matrix of the eight-vertex model.

There are well known symmetric solutions of the YBE (1.1) with the form of (1.3), namely the elliptic  $R$ -matrix of the  $XYZ$  model [4], the solutions with free fermionic property [13] (see also  $R$ -matrix for  $2d$  Ising Model (IM) in [12] and the three-parametric solutions to the YBE in [17]). But, although the YBE with the eight-vertex model's type  $4 \times 4$   $R$ -matrices are investigated and crucial solutions are obtained, however there is an open question remained here about the completeness of the solutions, as the authors usually take physically motivated symmetric matrices and the existed classification of the solutions reflects this fact. Here we present a full classification and the complete list of the solutions with the given form and also we show, that the YBE themselves are putting restrictions on the elements of  $R(u)$ , dictating some symmetry relations on them (namely, relations among the elements  $R_{kr}^{ij}$  and  $R_{1-k, 1-r}^{1-i, 1-j}$ ). Although we consider only the  $4 \times 4$ -matrices, but obviously such kind of behavior is valid also for the high dimensional matrices.

In the Section 2 we present consistency conditions for six- and eight-vertex type one-parametric  $R$ -matrices to be solutions to YBE, and a complete list of the nontrivial solutions. For both cases the compatibility conditions, expressed by the formulas (2.11, 2.12, 2.13) and (2.39, 2.40), can be proclaimed by means of two statements: *there are symmetric relations followed from YBE between the matrix elements and also the matrix elements obey expected homogeneous equations of the second order*. In the Section 3 the general parametrization for the general eight-vertex type solutions is given by means of trigonometric and elliptic functions. The solutions of the homogeneous eight-vertex model

obtained in [4] correspond to the case presented in the subsection 3.1. In the subsections 3.2 and 3.3 we present the general nonhomogeneous solutions. Next section is devoted to the description of the Hamiltonian operators of the corresponding 1d quantum spin-chain models.

In the Section 5, for completeness, we give the investigation of  $R$ -matrices, which we have excluded in the first two sections, since they contain more vanishing matrix elements than it is presented in (1.3), or have coinciding matrix elements, which in many cases leads to constant solutions. Investigations of the constant  $R$ -matrices are performed in the papers [19, 20]. We consider the spectral parameter dependent solutions, and it turns out, that along with the solutions, which can be obtained from the matrices considered in the Sections 2,3, after taking the appropriate limits, there are also independent solutions.

The Section 6 is devoted to the analysis of the tetrahedral Zamolodchikov's algebra [23] with the inhomogeneous  $R$ -matrices.

## 2 Investigation of the solutions to the YBE

In the consideration below we analyze Yang-Baxter equations for spin- $\frac{1}{2}$  systems in its most general form. These equations are well known, however we consider it is worthy to give one more time all the relations obviously for performing detailed analysis and for giving an exhaustive answer to the question what are the all one-spectral parameter dependent solutions in the form of (1.3) to the YBE (1.1).

We use the following notations for the matrix elements

$$\begin{aligned} R_{00}^{00} &= a_1(u), \quad R_{11}^{11} = a_2(u), \quad R_{01}^{01} = b_1(u), \quad R_{10}^{10} = b_2(u), \\ R_{01}^{10} &= c_1(u), \quad R_{10}^{01} = c_2(u), \quad R_{00}^{11} = d_1(u), \quad R_{11}^{00} = d_2(u). \end{aligned} \quad (2.1)$$

## 2.1 XXZ type $R$ -matrices

In the beginning let us briefly start with the case, when no creation or annihilation of the pairs may occur, if the  $R$ -matrix is assumed as particle's scattering matrix. It means that all the non-vanishing elements have the property  $i + j = k + r$ , together with (1.2). When the dimension of the states, where the  $R$ -matrix is acting, is two, it has form of the  $XXZ$  model's  $R$ -matrix, i.e.  $d_1 = d_2 = 0$  in (2.1). As we shall see later, the consideration of the general case ( $d_i \neq 0$ ) does not reflect all the peculiarities of this particular case (taken in the appropriate limit  $d_i \rightarrow 0$ ).

The YB equations in the matrix elements notations are given as follows

$$\sum_{j_1, j_2, j_3} R_{i_1 i_2}^{j_1 j_2}(u) R_{j_1 i_3}^{k_1 j_3}(u + w) R_{j_2 j_3}^{k_2 k_3}(w) = \sum_{j_1, j_2, j_3} R_{i_2 i_3}^{j_2 j_3}(w) R_{i_1 j_3}^{j_1 k_3}(u + w) R_{j_1 j_2}^{k_1 k_2}(u). \quad (2.1)$$

From the simple equations existing in (2.1) of this kind

$$c_1(u)c_1(w)c_2(u + w) = c_1(u + w)c_1(u)c_1(w), \quad (2.2)$$

it follows that  $c_1(u)/c_2(u)$  is an exponential function  $e^{u\alpha}$ , where  $\alpha$  is an arbitrary number. In the remaining 12 equations one can distinguish 6 pairs of the equations, such that in the each pair one of the equations will coincide with another, after using the relation (2.2). Taking into account this fact we choose the following six independent equations.

$$\begin{aligned} a_1(u + w)b_1(u)c_1(w) - a_1(u)b_1(u + w)c_1(w) + b_1(w)c_1(u + w)c_2(u) &= 0, \\ a_1(u)a_1(w)c_1(u + w) - b_1(w)b_2(u)c_1(u + w) - a_1(u + w)c_1(u)c_1(w) &= 0, \\ a_1(w)b_2(u + w)c_1(u) - a(u + w)b_2(w)c_1(u) - b_2(u)c_1(u + w)c_2(w) &= 0, \\ a_2(w)b_1(u + w)c_1(u) - a_2(u + w)b_1(w)c_1(u) - b_1(u)c_1(u + w)c_2(w) &= 0, \\ a_2(u)a_2(w)c_1(u + w) - b_1(u)b_2(w)c_1(u + w) - a_2(u + w)c_1(u)c_1(w) &= 0, \\ a_2(u + w)b_2(u)c_1(w) - a_2(u)b_2(u + w)c_1(w) + b_2(w)c_1(u + w)c_2(u) &= 0. \end{aligned} \quad (2.3)$$

The equations above are linear and homogeneous with respect of the functions  $a_1(w)$ ,  $b_1(w)$ ,  $c_1(w)$ ,  $a_2(w)$ ,  $b_2(w)$ ,  $c_2(w)$  and so have non-zero solutions, if the determinant of the matrix of the corresponding coefficients vanishes. The determinant reads as

$$\left( a_1(u)a_2(u + w)b_1(u + w)b_2(u) - a_1(u + w)a_2(u)b_1(u)b_2(u + w) \right) \quad (2.4)$$

$$\times (a_1(u)a_2(u) + b_1(u)b_2(u) - c_1(u)c_2(u))c_1(u)c_2(u)c_1(u+w)^4.$$

We are excluding the cases  $c_{1,2}(u) = 0$  by now and will observe that situation in the Section 5, which includes the exceptional cases.

**1. \*** The vanishing of the first bracket in the determinant's expression means

$$\frac{a_1(u)b_2(u)}{a_2(u)b_1(u)} = \text{constant} \equiv b_0. \quad (2.5)$$

Using this relation, from the first five equations presented above (2.3), we derive two possible relations. One of them is  $a_1(u) = b_1(u)\sqrt{\frac{-1}{b_0}}$ , which leads to the following subsequent equations:  $a_1(u) = a_2(u)$  &  $b_1(u)c_1(u+w)c_2(w) = 0$  (for such solutions see Section 5). The other relation implies

$$\frac{a_1(u)a_2(u) + b_1(u)b_2(u) - c_1(u)c_2(u)}{a_1(u)b_2(u)} = \text{constant} \equiv \Delta. \quad (2.6)$$

In this case (2.6) the remaining equations give the following additional requirements:

$$b_1(u) = b_0 b_2(u) \quad \& \quad a_1(u) = a_2(u). \quad (2.7)$$

**2. \*\*** The vanishing of the second bracket in the expression (2.4) means that the following relation holds

$$a_1(u)a_2(u) + b_1(u)b_2(u) - c_1(u)c_2(u) = 0. \quad (2.8)$$

If  $a_1(u) = a_2(u)$ , then from the equations (2.3) we find  $b_1(u) = b_2(u)$  **constant**, and this case becomes equivalent to the first discussed case (i.e. 2.5, 2.6) with  $\Delta = 0$ .

Let us observe now the situation, when  $a_1(u) \neq a_2(u)$ . From the first, third and forth equations in (2.3), taking into account that the change  $u \leftrightarrow v$  in the equations gives permissible and equivalent equations, we find relations

$$\frac{a_2(u) - a_1(u)}{b_1(u)} = \bar{\Delta} \quad \text{and} \quad \frac{b_1(u)}{b_2(u)} = b_0, \quad (2.9)$$

where  $\bar{\Delta}$  and  $b_0$  are constants. By means of the above relations (2.9) the constraint (2.8) takes the following form

$$a_1(u)^2 + b_1(u)^2 b_0 - c_1(u) c_2(u) = \bar{\Delta} a_1(u) b_1(u), \quad (2.10)$$

reminding the relation (2.6).

The possibilities **1** and **2** listed above are overlapping when  $\Delta = 0$  and  $\bar{\Delta} = 0$ .

**Let us summarize.** The general solutions of the YBE with  $R$ -matrix (2.1) have the following properties (we are omitting the arguments  $(u)$  of the functions  $f_i(u)$ ,  $f = a, b, c$  for simplicity)

$$c_1/c_2 = e^{\alpha u}, \quad (2.11)$$

$$b_1/b_2 = b_0 \quad (2.12)$$

$$\{a_1 = a_2 \ \& \ a_2 a_1 - c_1 c_2 + b_1 b_2 = a_1 b_1 \Delta\}^* \text{ or } \{a_2 - a_1 = \bar{\Delta} b_1 \ \& \ a_2 a_1 = c_1 c_2 - b_1 b_2\}^{**}. \quad (2.13)$$

Here  $\alpha, \Delta, \bar{\Delta}, b_0$  are arbitrary constants (they don't depend from the spectral parameter  $u$ ).

The most general solutions for two cases can be presented by means of the trigonometric parametrization (it is enough to choose  $a_1(u) = \sin[u + u_0]$  and  $c_1(u) = e^{\frac{\alpha u}{2}} \sin[u_0]$ , as for the XXZ  $R$ -matrix, then the expressions for the functions  $b_i(u)$ ,  $c_2(u)$  and  $a_2(u)$  will follow from (2.11, 2.12, 2.13))

$$R^{*/**}(u) = \begin{pmatrix} \sin[u + u_0] & 0 & 0 & 0 \\ 0 & \sin[u] \sqrt{b_0} & e^{\frac{\alpha u}{2}} \sin[u_0] & 0 \\ 0 & e^{-\frac{\alpha u}{2}} \sin[u_0] & \sin[u] \frac{1}{\sqrt{b_0}} & 0 \\ 0 & 0 & 0 & \sin[u_0] \pm u \end{pmatrix}. \quad (2.14)$$

Any other parametrization can be obtained from this one either by the redefinitions of the arguments or by basis change. The rational solutions correspond to the limit  $\lim_{h \rightarrow 0} R(hu) / \sin[hu]$ .

The first case  $*$  in (2.13) corresponds to the ordinary XXZ model (intertwiner matrix of the  $sl_q(2)$  algebra at general  $q$ ). The second logarithmic derivative of the transfer matrix constructed by this matrix gives the Hamiltonian  $H$  of the anisotropic Heisenberg magnetic with anisotropy parameter  $\Delta/2 = \cos[u_0]$ .

If we represent  $R$  by means of the tensor products of the Pauli  $\sigma$  matrices, then  $H$  becomes (we take  $b_0 = 1, \alpha = 0$ )

$$H^* = \sum_k \left( \sigma_1(k)\sigma_1(k+1) + \sigma_2(k)\sigma_2(k+1) + \frac{\Delta}{2}\sigma_3(k)\sigma_3(k+1) \right). \quad (2.15)$$

The second case  $(**)$  corresponds to the  $XX$ -model in the transverse magnetic field  $h = \bar{\Delta}/2 = \cos[u_0]$  (intertwiner matrix of the  $sl_q(2)$  matrix at  $q = i$ , nilpotent [14] and cyclic irreps [21]).

$$H^{**} = \sum_k \left( \sigma_1(k)\sigma_1(k+1) + \sigma_2(k)\sigma_2(k+1) + \bar{\Delta}\sigma_3(k) \right). \quad (2.16)$$

We see, that the only significantly different solution from the known  $XXZ$ -model's  $R$ -matrix, suggests  $R_{11}^{11} \neq R_{00}^{00}$ . The appearance of the function  $e^{\alpha u}$  can be addressed to the change of the basis vectors of the definition spaces.

## 2.2 YBE with extended XYZ type $R$ -matrices

If one allow the creation or annihilation of the pairs in the scattering matrix (eight-vertex model), we must deal with the more general form of the  $R$ -matrix (2.1). The simplest equations among the YBE (2.1) are

$$c_1(u)c_1(w)c_2(u+w) - c_1(u+w)c_2(u)c_2(w) = 0, \quad (2.17)$$

$$-c_1(u+w)d_1(w)d_2(u) + c_2(u+w)d_1(u)d_2(w) = 0, \quad (2.18)$$

$$c_1(u)d_1(w)d_2(u+w) - c_2(u)d_1(u+w)d_2(w) = 0. \quad (2.19)$$

Let  $c_i \neq 0$ . The equation (2.17) leads to  $c_1(u) = c_2(u)e^{ku}$ , with arbitrary constant  $k$ . Placing this relation into the equations (2.18, 2.19), we come to  $k = 0$  (*note*, that when



$d_i = 0$ ,  $k$  is arbitrary, see subsection 2.1). Then  $d_2(u) = d_0 d_1(u)$ ,  $d_0$  is an arbitrary constant. So we have

$$c_1(u) = c_2(u), \quad d_2(u) = d_0 d_1(u). \quad (2.20)$$

From the analysis of the previous case with matrix (2.1) we learn, that there are two different non trivial restrictions on the solutions to YBE. One corresponds to the case  $a_1(u) = a_2(u)$ , the second one to  $a_1(u) \neq a_2(u)$ .

1. \* Let us at first consider the case  $a_1(u) = a_2(u)$ .

Comparing the following equations from the set of the YBE (2.1)

$$a_1(w)(c_1(u+w)d_1(u) - c_1(u)d_1(u+w)) + (a_1(u)a_1(u+w) - b_1(u)b_1(u+w))d_1(w) = 0, \quad (2.21)$$

$$a_1(w)(c_1(u+w)d_1(u) - c_1(u)d_1(u+w)) + (a_1(u)a_1(u+w) - b_2(u)b_2(u+w))d_1(w) = 0 \quad (2.22)$$

we immediately find

$$b_2(u) = b_0 b_1(u), \quad b_0^2 = 1. \quad (2.23)$$

Note, that when  $d_i = 0$ ,  $b_0$  is an arbitrary constant.

Taking into account (2.20) and (2.23), there are only six independent equations in the YBE (quite similarly to the XYZ case, discussed in the work [4]).

$$\begin{aligned} & a(w)c(u+w)d(u) - a(w)c(u)d(u+w) + a(u)a(u+w)d(w) - b(u)b(u+w)d(w) = 0; \\ & -b(w)c(u)c(u+w) - a(u+w)b(u)c(w) + a(u)b(u+w)c(w) + b_0 d_0 b(w)d(u)d(u+w) = 0; \\ & a(u)a(w)c(u+w) - b_0 b(u)b(w)c(u+w) - a(u+w)c(u)c(w) + d_0 a(u+w)d(u)d(w) = 0; \\ & a(w)b_0 b(u+w)c(u) - b_0(a(u+w)b(w)c(u) + b(u)c(u+w)c(w)) + d_0 b(u)d(u+w)d(w) = 0; \\ & b_0 b(u+w)c(w)d(u) - b_0 a(w)b(u)d(u+w) + a(u)b(w)d(u+w) - b(u+w)c(u)d(w) = 0; \\ & -a(u+w)a(w)d(u) + b(u+w)b(w)d(u) + a(u)c(w)d(u+w) - a(u)c(u+w)d(w) = 0. \end{aligned} \quad (2.24)$$

In the given equations we are omitting the index 1, taking  $f_1(u) \equiv f(u)$ ,  $f = a, b, c, d$ .

The consistency condition of the last four equations is

$$\begin{aligned} & \left( a(u+w)b(u+w)c(u)d(u) - b_0a(u)b(u)c(u+w)d(u+w) \right) \times \\ & \left( (a^2(u) - b^2(u))(c^2(u+w) - b_0d_0d^2(u+w)) + (a^2(u+w) - b^2(u+w))(b_0c^2(u) - d_0d^2(u)) \right) = 0. \end{aligned} \quad (2.25)$$

As we see there are two cases that one must observe.

**1.1** If the first bracket in (2.25) vanishes,

$$a(u+w)b(u+w)c(u)d(u) - b_0a(u)b(u)c(u+w)d(u+w) = 0, \quad (2.26)$$

the following relations are true

$$\frac{a(u)b(u)}{c(u)d(u)} = \text{constant}, \quad b_0 = 1. \quad (2.27)$$

This is identical to the case, considered in [4] with homogeneous and symmetric  $R$ -matrix.

As it is known, the remaining equations give the constraint

$$\frac{a^2(u) + b^2(u) - c^2(u) - d^2(u)}{a(u)b(u)} = \text{constant}. \quad (2.28)$$

This is the well observed case (see [4]) of the XYZ model's  $R$ -matrix (which will be given in a precise form in the next section) satisfying this constraint.

**1.2** The vanishing of the second bracket in (2.25)

$$\left( a^2(u) - b^2(u) \right) \left( c^2(u+w) - b_0d_0d^2(u+w) \right) + \left( a^2(u+w) - b^2(u+w) \right) \left( b_0c^2(u) - d_0d^2(u) \right) = 0,$$

gives the relations

$$\frac{a^2(u) - b^2(u)}{b_0c^2(u) - d_0d^2(u)} = \text{constant}, \quad b_0 = -1. \quad (2.29)$$

The constant in (2.29) can be fixed from the analysis of the remaining equations, or by a rather simple way. Let us fix the variable  $w$  to be 0, then from the set of the YBE

we can see, that non-trivial ( $a(u) \neq \pm b(u)$ ,  $d(u) \neq \pm c(u)$ ) solutions demand  $a(0) = c(0)$  and  $b(0) = d(0) = 0$ . This fixes the constant to be  $-1$ . So

$$a^2(u) - b^2(u) - c^2(u) - d_0 d^2(u) = 0. \quad (2.30)$$

A particular case of (2.29) is (below  $x_1, x_2$  are constant numbers)

$$a^2(u) - b^2(u) = x_1, \quad c^2(u) + d_0 d^2(u) = x_2, \quad b_0 = -1. \quad (2.31)$$

Analyzing the equations (2.24) we find that  $x_1 = 0$ ,  $x_2 = 0$ . Such exceptional cases will be discussed ( $f_i(u) = 0$  or  $f(u) = \pm g(u)$ ,  $f, g = a, b, c, d$ ) in the Section 5.

**2.\*\*** Now let us consider the case  $a_1(u) \neq a_2(u)$ .

The YBE now contain twelve independent equations.

$$\begin{aligned} a_1(u)b_1(u+w)c_1(w) - b_1(w)c_1(u)c_1(u+w) - a_1(u+w)b_1(u)c_1(w) + d_0b_2(w)d_1(u)d_1(u+w) &= 0, \\ a_1(u)a_1(w)c_1(u+w) - b_1(w)b_2(u)c_1(u+w) - a_1(u+w)c_1(u)c_1(w) + a_2(u+w)d_0d_1(u)d_1(w) &= 0, \\ a_1(w)b_2(u+w)c_1(u) - a_1(u+w)b_2(w)c_1(u) - b_2(u)c_1(u+w)c_1(w) + d_0b_1(u)d_1(u+w)d_1(w) &= 0, \\ a_2(u+w)b_1(w)c_1(u) + b_1(u)c_1(u+w)c_1(w) - d_0b_2(u)d_1(u+w)d_1(w) - a_2(w)b_1(u+w)c_1(u) &= 0, \\ b_1(u)b_2(w)c_1(u+w) + a_2(u+w)c_1(u)c_1(w) - a_2(u)a_2(w)c_1(u+w) - d_0a_1(u+w)d_1(u)d_1(w) &= 0, \\ a_2(u)b_2(u+w)c_1(w) + d_0b_1(w)d_1(u)d_1(u+w) - b_2(w)c_1(u)c_1(u+w) - a_2(u+w)b_2(u)c_1(w) &= 0, \\ a_2(w)c_1(u+w)d_1(u) - a_1(w)c_1(u)d_1(u+w) + a_1(u)a_1(u+w)d_1(w) - b_1(u)b_1(u+w)d_1(w) &= 0, \\ b_2(u+w)c_1(w)d_1(u) + a_1(u)b_1(w)d_1(u+w) - a_1(w)b_2(u)d_1(u+w) - b_1(u+w)c_1(u)d_1(w) &= 0, \\ b_2(u+w)b_2(w)d_1(u) + a_1(u)c_1(w)d_1(u+w) - a_2(u)c_1(u+w)d_1(w) - a_1(u+w)a_1(w)d_1(u) &= 0, \\ a_2(u+w)a_2(w)d_1(u) - b_1(u+w)b_1(w)d_1(u) - a_2(u)c_1(w)d_1(u+w) + a_1(u)c_1(u+w)d_1(w) &= 0, \\ a_2(w)b_1(u)d_1(u+w) - a_2(u)b_2(w)d_1(u+w) + b_2(u+w)c_1(u)d_1(w) - b_1(u+w)c_1(w)d_1(u) &= 0, \\ a_2(w)c_1(u)d_1(u+w) - a_2(u)a_2(u+w)d_1(w) + b_2(u)b_2(u+w)d_1(w) - a_1(w)c_1(u+w)d_1(u) &= 0. \end{aligned} \quad (2.32)$$

Let us compare the third equation in the set (2.32) with the sixth equation, after interchanging the variables  $u$  and  $w$  in the last one. Then taking out one from the other, we come to (assuming, that  $c(u) \neq 0$ )

$$\frac{a_1(w) - a_2(w)}{b_2(w)} = \frac{a_1(u+w) - a_1(u+w)}{b_2(u+w)}. \quad (2.33)$$

Now we can interchange the variables  $u$  and  $w$  in the fourth equation in (2.32) and remove it from the first equation. As a result the following relation holds

$$\frac{a_1(u) - a_2(u)}{b_1(u)} = \frac{a_1(u+w) - a_1(u+w)}{b_1(u+w)}. \quad (2.34)$$

So, the following two constraints are followed from the equations (2.33, 2.34)

$$\frac{a_1(u) - a_2(u)}{b_1(u)} = \bar{\Delta}, \quad b_2(u) = b_0 b_1(u), \quad (2.35)$$

where  $\bar{\Delta}$  and  $b_0$  are constants. From the first and sixth equations it follows, that  $(-1 + b_0^2)b_1(w)d_0d_1(u)d_1(u+w) = 0$ . When the elements of the  $R$ -matrix are not 0, then the relation  $b_0^2 = 1$  is true. Taking into account this, from the eighth and eleventh equations, we find  $b_1(u)d_1(u+w)(-1 + b_0)b(w)\bar{\Delta} = 0$ . If  $\bar{\Delta} \neq 0$ , i.e.  $a_1(u) \neq a_2(u)$ , then

$$b_0 = 1. \quad (2.36)$$

The independent equations in (2.32) now are nine, as one can neglect fourth, sixth and eleventh equations, which are contained in the remaining ones. Now let us consider the following four equations: the first equation in (2.32), the second one, the difference of the ninth and tenth equations and the difference of the seventh and twelfth ones. The consistency condition for the solutions of these linear equations in respect of  $a_1(w)$ ,  $b_1(w)$ ,  $c_1(w)$ ,  $d_1(w)$  reads as

$$\begin{aligned} & \left( a_1(u+w)b_1(u) - a_1(u)b_1(u+w) \right) \left( a_1(u+w)b_1(u) + b_1(u+w)(a_1(u) + b_1(u)\bar{\Delta}) \right) \\ & \times (-1 + a_1^2(u) + b_1^2(u) - d_0d_1^2(u) + a_1(u)b_1(u)\bar{\Delta}) = 0. \end{aligned} \quad (2.37)$$

The vanishing of the first or the second brackets brings to constant solutions. The equality of the expression in the third bracket to zero presents nontrivial constraint on the solutions

$$a_1^2(u) + b_1^2(u) - c_1^2(u) - d_0d_1^2(u) = -a_1(u)b_1(u)\bar{\Delta}. \quad (2.38)$$

**Summary of this subsection.** The constraints, laid by the YBE (2.1) on the  $R(u)$ -matrix (2.1) are of the following form

$$\begin{aligned} c_1/c_2 &= 1, \\ b_2/b_1 &= b_0 \quad \& \quad b_0^2 = 1, \quad d_2/d_1 = d_0, \end{aligned} \quad (2.39)$$

$$\begin{aligned} &\left\{ a_1/a_2 = 1 \quad \& \quad \begin{aligned} &b_0 = 1, \quad a_1 b_1 = c_1 d_1 x, \quad a_2 a_1 - c_1 c_2 + b_1 b_2 - d_1 d_2 = 2a_1 b_1 \Delta \\ &b_0 = -1, \quad a_1 a_2 + b_1 b_2 - c_1 c_2 - d_1 d_2 = 0, \end{aligned} \right\}^* \\ \text{or} \quad &\left\{ b_0 = 1, \quad a_2 - a_1 = \bar{\Delta} b_1 \quad \& \quad a_2 a_1 - d_1 d_2 = c_1 c_2 - b_1 b_2 \right\}^{**}. \end{aligned} \quad (2.40)$$

Here, as it was used before, by the parameters  $b_0$ ,  $d_0$ ,  $x$ ,  $\Delta$ ,  $\bar{\Delta}$  we have denoted the constants. As we have noted, when the functions  $d_i \neq 0$ , then the parameter  $b_0$  is not arbitrary.

In the following sections we shall give a rather detailed analysis of the obtained constraints and discuss the possible parameterizations of the elements of  $R$ -matrices, beginning from the well known  $XYZ$ -model case. The corresponding 1d spin-chain Hamiltonian operators will be discussed as well.

### 3 General parametrization of the solutions

The obtained relations on the matrix elements suggest that it is possible to write down different equivalent parameterizations of the  $R$ -matrices by means of the Jacobi elliptic functions (as it is well known for the  $XYZ$  model matrix  $R_{XYZ}$ ) and trigonometric functions [4, 14, 15]. This can be achieved by choosing  $c(u) = 1$  (normalization) and fixing the function  $b(u)$ . Then determining the other elements from the obtained constraints and YBE one can find out the whole solution.

### 3.1 $R_{XYZ}$ -matrix: $a_1 = a_2$ and $b_1 = b_2$

At first we analyze the equations (2.24) with constraints (2.27) and (2.28), corresponding to the case in \* (2.40) for which  $b_0 = 1$ . For definiteness we take  $d_0 = 1$ , as after finding the function  $d(u)$ , we can put  $d_1(u) = e^{\gamma/2}d(u)$  and  $d_2(u) = e^{-\gamma/2}d(u)$ .

Setting  $w = -u$  in the two first equations of (2.24), we find

$$\frac{d(u)}{a(u)} = -\frac{d(-u)}{a(-u)}, \quad \frac{b(u)}{c(u)} = -\frac{b(-u)}{c(-u)}. \quad (3.1)$$

So,  $b(u)/c(u)$  and  $d(u)/a(u)$  are odd functions. The third equation gives

$$a(u)a(-u) - b(u)b(-u) - c(u)c(-u) + d(u)d(-u) = 0, \quad (3.2)$$

which, taking into account the previous relations (3.1), can be rewritten as

$$([a(u)]^2 - [d(u)]^2) \frac{d(-u)}{d(u)} = ([b(u)]^2 - [c(u)]^2) \frac{c(-u)}{c(u)}. \quad (3.3)$$

Taking  $c(u) = 1$ , from the relation  $a(0) = c(0)$  found in the previous section, it follows  $a(0) = 1$ . If there is a point  $u_0$ , where  $a(u_0) = 0$ , then the constraint (2.27) gives  $d(u_0) = 0$ . Putting  $u = u_0$  in the second equation of the set (2.24) we can find

$$a(w + u_0) = -b(w)/b(u_0) \quad \text{or} \quad a(w) = -b(w - u_0)/b(u_0). \quad (3.4)$$

The relation (2.28) imposes  $b(u_0) = \pm c(u_0) = \pm 1$ . One can fix  $b(u) = \text{sn}[u, k]/\text{sn}[\lambda, k]$ , as the YB equations reading now as transformation equations of the function  $b(u)$  coincide with those of the Jacobi elliptic functions [4]. To the same conclusion one can come analyzing the differential equations for the function  $b(u)$  which can be obtained from YBE. So, there are two possibilities  $u_0 = \pm\lambda$  and  $a(u) = -\text{sn}[u - \lambda, k]/\text{sn}[\lambda, k]$  or  $a(u) = \text{sn}[u + \lambda, k]/\text{sn}[\lambda, k]$ . Let us fix  $a(u) = \text{sn}[u + \lambda, k]/\text{sn}[\lambda, k]$ , as the other case can be obtained from this one by transformations  $u \rightarrow -u$  or  $\lambda \rightarrow -\lambda$ . The function  $d(u)$  from the constraint (2.27) takes the form

$$d(u) = \text{constant } a(u)b(u) = \text{constant } \text{sn}[u + \lambda, k]\text{sn}[u, k]/(\text{sn}[\lambda, k])^2. \quad (3.5)$$

From the equation (3.3), after putting  $u = -\lambda$ , one gets that the value of the above constant  $= \pm k(\text{sn}[\lambda, k])^2$ . The interchange  $k \leftrightarrow -k$  leaves the Jacobi-elliptic functions invariant. For definiteness we choose

$$d(u) = k \text{sn}[u + \lambda, k] \text{sn}[u, k]. \quad (3.6)$$

So, in general, the homogeneous  $R$ -matrix of the eight-vertex (or  $XYZ$ ) model can be parameterized by two model parameters,  $k$  and  $\lambda$ , as follows

$$R_{xyz}(u) = \begin{pmatrix} \frac{\text{sn}[u+\lambda, k]}{\text{sn}[\lambda, k]} & 0 & 0 & e^{\gamma/2} k \text{sn}[\lambda + u, k] \text{sn}[u, k] \\ 0 & \frac{\text{sn}[u, k]}{\text{sn}[\lambda, k]} & 1 & 0 \\ 0 & 1 & \frac{\text{sn}[u, k]}{\text{sn}[\lambda, k]} & 0 \\ e^{-\gamma/2} k \text{sn}[\lambda + u, k] \text{sn}[u, k] & 0 & 0 & \frac{\text{sn}[u+\lambda, k]}{\text{sn}[\lambda, k]} \end{pmatrix}. \quad (3.7)$$

Then the constant in (2.28) can be written as  $2\text{cn}[\lambda, k]\text{dn}[\lambda, k]$ .

Any other parameterizations can be obtained by changing the function  $b(u)$ , which can be regarded as a replacement of the spectral parameter. Two different parameterizations by elliptic functions can be connected one with another by the transformation rules of the elliptic functions [15].

The "free-fermionic" case, when constant in (2.28) is zero, corresponds to the  $XY$ -model. It takes place, say when  $\lambda = K(k)$ , where  $K(k)$  is the complete elliptic integral of the first kind. The corresponding  $R_{XY}(u)$  matrix has the form

$$R_{xy}(u) = \begin{pmatrix} \frac{\text{cn}[u, k]}{\text{dn}[u, k]} & 0 & 0 & e^{\gamma/2} k \frac{\text{cn}[u, k] \text{sn}[u, k]}{\text{dn}[u, k]} \\ 0 & \text{sn}[u, k] & 1 & 0 \\ 0 & 1 & \text{sn}[u, k] & 0 \\ e^{-\gamma/2} k \frac{\text{cn}[u, k] \text{sn}[u, k]}{\text{dn}[u, k]} & 0 & 0 & \frac{\text{cn}[u, k]}{\text{dn}[u, k]} \end{pmatrix}. \quad (3.8)$$

As we see, for the extended case of  $R_{XYZ}$  with  $a_1(u) = a_2(u)$  the only new parameter arises due to  $d_0$ . Something essentially new arises for the inhomogeneous cases ( $b_1(u) \neq b_2(u)$  or  $a_1(u) \neq a_2(u)$ ).

### 3.2 Free fermionic solution with $a_1 = a_2$ and $b_1 = -b_2$

Here we would like to consider the case in \* (2.40) for which  $b_0 = -1$ . It is a free fermionic case.

There are two possible solutions for this case. One of them is a rather trivial and demands  $a(u) = \pm b(u)$ . It gives

$$R(u) = c(u) \begin{pmatrix} e^{\alpha u} & 0 & 0 & e^{\gamma} \\ 0 & \pm e^{\alpha u} & 1 & 0 \\ 0 & 1 & \mp e^{\alpha u} & 0 \\ -e^{-\gamma} & 0 & 0 & e^{\alpha u} \end{pmatrix}. \quad (3.9)$$

$\alpha, \gamma$  are arbitrary numbers (see also Section 5).

The second solution can be constructed by this way. We suppose  $b(0) = 0$ . Here we can choose  $c(u) = 1$  as in the previous,  $XYZ$ , case. From the independent YB equations we can obtain  $a(0) = 1$ . The constraint (2.30) gives  $d(0) = 0$  (and also  $a'(0) = 0$ ). Expansion of the first equation in (2.24) near the point  $w = 0$  brings to a simple differential equation for function  $d(u)$ , solution of which is

$$d(u) = \tan[\alpha u] / \sqrt{d_0}. \quad (3.10)$$

From the next two equations, using  $a'(0) = 0$ , we get  $a(u) + b(u) = e^{(\varepsilon u)} \sec[\alpha u]$  ( $\alpha$  and  $\varepsilon$  are arbitrary numbers). Then it is easy to find that  $b(u) = \sinh[\varepsilon u] \sec[\alpha u]$  and  $a(u) = \cosh[\varepsilon u] \sec[\alpha u]$ .

So, the matrix form of this solution looks like as follows

$$\tilde{R}(u) = \begin{pmatrix} \cosh[\varepsilon u] \sec[\alpha u] & 0 & 0 & e^{\gamma/2} \tan[\alpha u] \\ 0 & \sinh[\varepsilon u] \sec[\alpha u] & 1 & 0 \\ 0 & 1 & -\sinh[\varepsilon u] \sec[\alpha u] & 0 \\ e^{-\gamma/2} \tan[\alpha u] & 0 & 0 & \cosh[\varepsilon u] \sec[\alpha u] \end{pmatrix} \quad (3.11)$$



We have replaced  $d_0$  by  $e^\gamma$ .

As the numbers  $\alpha$  and  $\varepsilon$  are arbitrary, we can take the matrix  $\tilde{R}(u)$  (3.11) as a matrix which has two spectral parameters  $\tilde{R}(u; v)$  (after multiplication by  $\cos[v]$ )

$$\tilde{R}(u; v) = \begin{pmatrix} \cosh[u] & 0 & 0 & e^{\gamma/2} \sin[v] \\ 0 & \sinh[u] & \cos[v] & 0 \\ 0 & \cos[v] & -\sinh[u] & 0 \\ e^{-\gamma/2} \sin[v] & 0 & 0 & \cosh[u] \end{pmatrix}. \quad (3.12)$$

and satisfies to the following YB equations

$$\begin{aligned} \sum_{j_1, j_2, j_3} R_{i_1 i_2}^{j_1 j_2}(u; v) R_{j_1 i_3}^{k_1 j_3}(u + w; v + y) R_{j_2 j_3}^{k_2 k_3}(w; y) = \\ \sum_{j_1, j_2, j_3} R_{i_2 i_3}^{j_2 j_3}(w; y) R_{i_1 j_3}^{j_1 k_3}(u + w; v + y) R_{j_1 j_2}^{k_1 k_2}(u; v). \end{aligned} \quad (3.13)$$

Note, that this free-fermionic solution is not followed from the tree-parametric elliptic solutions presented in [17].

### 3.3 The nonhomogeneous case with $a_1 \neq a_2$ .

Let us proceed further and write down the solutions for the inhomogeneous case  $a_1 \neq a_2$ . In the previous cases the solutions for the equations (2.24) have been found, using the necessary conditions (\*2.40) and the properties of the Jakobi elliptic functions [4]. In the same way one can try to find appropriate parametrization for the equations (2.32), taking into account the constraints (2.35, 2.36, 2.38) (i.e. (\*\*2.40)).

The analysis of the equations gives that  $b_1(0) = 0$ ,  $a_1(0) = c_1(0)$ ,  $d_1(0) = 0$  (elsewise we shall have constant solutions). The first equation implies that  $b_1(u)$  is an odd function. Using it, from the second and fifth equations, placing  $w = -u$ , we find  $a_2(u) = a_1(-u)$ , so

$$a_1(-u) = a_1(u) + \bar{\Delta} b_1(u).$$

If there is a point  $u_0$ , where  $b(u_0) = 1$ , then we can use it, for parameterizing the remaining functions. We can choose as in the previous cases  $b(u) = \text{sn}[u, k]/\text{sn}[u_0, k]$ .

The first equation in (2.32), after differentiating near the point  $w = 0$ , gives an expression for the function  $a(u) \equiv a_1(u)$  via the functions  $b(u)$ ,  $d(u)$  and their derivatives.

$$a(u) = \frac{1}{b'(0)} (b(u)a'(0) + b'(u) + b(u)d(u)d'(0)), \quad b''(0) = 0. \quad (3.14)$$

From the relation

$$a(u)a(-u) + b(u)^2 - 1 - d(u)^2 = 0, \quad (3.15)$$

after expansion it at the point  $u = 0$ , we can find  $a'(0) = -\bar{\Delta}b'(0)/2$ . Then placing the above expression of the function  $a(u)$  into the equation (3.15), we find two possible solutions for the function  $d(u)$ . From the same equation it follows also the following relation

$$\bar{\Delta}^2 - 4 + 4(\text{sn}[u_0, k])^2(1 + k - d'(0)^2) = 0.$$

As we can find from the analysis of the equations (seventh equation in (2.32)),  $d(u)$  is an odd function, which gives the following constraint

$$(1 - d'(0)^2)(k^2 - d'(0)^2) = 0.$$

If we take  $d'(0)^2 = 1$ , then we come to the following parametrization

$$a(u) = \frac{\text{dn}[u, k]}{\text{cn}[u, k]} - \frac{\bar{\Delta}\text{sn}[u, k]}{2\text{sn}[u_0, k]}, \quad (3.16)$$

$$d(u) = \frac{\pm \text{dn}[u, k]\text{sn}[u, k]}{\text{cn}[u, k]}, \quad (3.17)$$

$$\bar{\Delta}^2 = 4(\text{dn}[u_0, k])^2. \quad (3.18)$$

The corresponding  $R(u)$ -matrix looks like

$$R_1(u, k) = \begin{pmatrix} \frac{\text{dn}[u, k]}{\text{cn}[u, k]} \pm \frac{\text{dn}[u_0, k]\text{sn}[u, k]}{\text{sn}[u_0, k]} & 0 & 0 & e^{\gamma/2} \frac{\text{dn}[u, k]\text{sn}[u, k]}{\text{cn}[u, k]} \\ 0 & \frac{\text{sn}[u, k]}{\text{sn}[u_0, k]} & 1 & 0 \\ 0 & 1 & \frac{\text{sn}[u, k]}{\text{sn}[u_0, k]} & 0 \\ e^{-\gamma/2} \frac{\text{dn}[u, k]\text{sn}[u, k]}{\text{cn}[u, k]} & 0 & 0 & \frac{\text{dn}[u, k]}{\text{cn}[u, k]} \mp \frac{\text{dn}[u_0, k]\text{sn}[u, k]}{\text{sn}[u_0, k]} \end{pmatrix} \quad (3.19)$$

When we choose  $d'(0)^2 = k^2$ , then

$$a(u) = \frac{\text{cn}[u, k]}{\text{dn}[u, k]} - \frac{\bar{\Delta} \text{sn}[u, k]}{2 \text{sn}[u_0, k]}, \quad (3.20)$$

$$d(u) = \frac{\pm k \text{cn}[u, k] \text{sn}[u, k]}{\text{dn}[u, k]}, \quad (3.21)$$

$$\bar{\Delta}^2 = 4(\text{cn}[u_0, k])^2. \quad (3.22)$$

$$R_2(u, k) = \begin{pmatrix} \frac{\text{cn}[u, k]}{\text{dn}[u, k]} \pm \frac{\text{cn}[u_0, k] \text{sn}[u, k]}{\text{sn}[u_0, k]} & 0 & 0 & e^{\gamma/2} k \frac{\text{cn}[u, k] \text{sn}[u, k]}{\text{dn}[u, k]} \\ 0 & \frac{\text{sn}[u, k]}{\text{sn}[u_0, k]} & 1 & 0 \\ 0 & 1 & \frac{\text{sn}[u, k]}{\text{sn}[u_0, k]} & 0 \\ e^{-\gamma/2} k \frac{\text{cn}[u, k] \text{sn}[u, k]}{\text{dn}[u, k]} & 0 & 0 & \frac{\text{cn}[u, k]}{\text{dn}[u, k]} \mp \frac{\text{cn}[u_0, k] \text{sn}[u, k]}{\text{sn}[u_0, k]} \end{pmatrix} \quad (3.23)$$

These two solutions are connected one with another by the so called "transformations of the first degree" of the elliptic functions:  $k \rightarrow 1/k$ ,  $u \rightarrow ku$  (see Table 21.6 – 8 in [16]).

If  $u_0 = K(k)$ , where  $K$  is the complete elliptic integral of the first kind with the module  $k$ , then the first version  $R_1$  corresponds exactly to the  $R$ -matrix, which we have found in [12] for  $2d$  IM. The second solution  $R_2$  at  $u_0 = K(k)$  corresponds to the  $XY$ -model matrix  $R_{XY}$  (3.8). Also, one can check, that the three-parametric solution  $R(u; \psi_1, \psi_2)$  in [17] coincides with (3.23) up to renormalization of the  $R$ -matrix by an overall function (i.e. by function  $c(u; \psi_1, \psi_2)$ ) when the following relations have been fulfilled  $\psi_1 = \psi_2 = u_0$ ,  $u \rightarrow 2u$ .

All the obtained  $R(u)$  matrices, (3.7), (3.11), (3.19) and (3.23), after an appropriate normalization (multiplication by a function) have a property of unitarity

$$R(u)[R(-u)]^+ = I. \quad (3.24)$$

The discussed inhomogeneous matrices (3.9), (3.11), (3.19) and (3.23) have the "free-fermionic" property:  $a_1 a_2 + b_1 b_2 = c_1 c_2 + d_1 d_2$ . A solution, which has no such property, mets amongst the exceptional cases, discussed in the Section 5 (5.18). The symmetry (quite similar to the case of the  $sl_q(2)$  symmetry of the  $XXZ$  model) of the inhomogeneous trigonometric solutions with the "free-fermionic" property is unveiled in the article [21].

## 4 The corresponding quantum one dimensional Hamiltonian operators

In the article [12] the authors have presented the  $R$ -matrix for two-dimensional IM [8]. The main idea there was to find the matrix which satisfies YBE just from the Boltzmann weights of the 2d IM with coupling constants  $J_1, J_2$ . Then using the Baxter's transformation [4]

$$e^{\pm 2J_1} = \text{cn}(i u, k) \pm i \text{sn}(i u, k), \quad (4.1)$$

$$e^{\pm 2J_2} = i(\text{dn}(i u, k) \pm 1)/(k \text{sn}(i u, k)). \quad (4.2)$$

and after some unitary transformation, the  $4 \times 4$ -matrix, corresponding to the Boltzmann weight of the 2d IM, takes the form

$$R(u, k) = \begin{pmatrix} \text{cn}[i u, k] \text{dn}[i u, k] - i k \text{sn}[i u, k] & 0 & 0 & i \text{sn}[i u, k] \text{dn}[i u, k] \\ 0 & -i \text{sn}[i u, k] & \text{cn}[i u, k] & 0 \\ 0 & \text{cn}[i u, k] & -i \text{sn}[i u, k] & 0 \\ i \text{sn}[i u, k] \text{dn}[i u, k] & 0 & 0 & \text{cn}[i u, k] \text{dn}[i u, k] + i k \text{sn}[i u, k] \end{pmatrix}. \quad (4.3)$$

Note that

$$R(0, k) = P, \quad (4.4)$$

where  $P$  is the permutation matrix. We have verified in [12] that  $R(u, k)$  satisfies the Yang-Baxter equation.

If one performs the transformation called by Akhiezer as the "second main transformation of the first degree" [15] (also, see [16]):

$$\text{sn}(i u, k) = i \frac{\text{sn}(u, k')}{\text{cn}(u, k')}, \quad \text{cn}(i u, k) = \frac{1}{\text{cn}(u, k')}, \quad \text{dn}(i u, k) = \frac{\text{dn}(u, k')}{\text{cn}(u, k')}, \quad (4.5)$$

in (4.3), then it becomes equivalent to the matrix (3.19) with  $u_0 = K$ .

The Hamiltonian operators of the one-dimensional quantum spin-chain models, corresponding to this  $R$ -matrix and the matrix of the XY-model, which also describes free

fermions, are different. The "check" version of the matrix (4.3)  $\check{R} = RP$  in the limit of small  $u$ -s or

$$J_1 \sim J\Delta t, \quad e^{-J_2} \sim h\Delta t, \quad \Delta t \ll 1, \quad (4.6)$$

can be written in this operator form  $\mathbf{R}$  by using  $\sigma_i$ -matrices

$$\mathbf{R}_{IM} = \frac{1}{h\Delta t} (1 \otimes 1 + 2\Delta t (J\sigma_1 \otimes \sigma_1 + h(1 \otimes \sigma_z + \sigma_z \otimes 1))). \quad (4.7)$$

While the  $\check{R}$ -matrix of the XY-model has not the term  $h(1 \otimes \sigma_z + \sigma_z \otimes 1)$  corresponding to the interaction with transverse magnetic field, and has the following expansion

$$\mathbf{R}_{XY} = (1 \otimes 1 + u(\mathbf{J}_1\sigma_1 \otimes \sigma_1 + \mathbf{J}_2\sigma_2 \otimes \sigma_2)). \quad (4.8)$$

Therefore the first one describes the quantum 1d Ising model in the transverse field  $h$ ,

$$H_I = \sum_i (J\sigma_1[i]\sigma_1[i+1] + h\sigma_z[i]), \quad (4.9)$$

and the R-matrix of the eight-vertex model (3.7), with the free-fermionic condition, (3.8), describes the quantum 1d XY-model.

$$H_{XY} = \sum_i (J_1\sigma_1[i]\sigma_1[i+1] + J_2\sigma_2[i]\sigma_2[i+1]). \quad (4.10)$$

What is then the meaning of the new parameter  $u_0$ ? One can verify, that this parameter characterizes the parameters  $J_1, J_2$  of the XY model in the transverse field. The matrices  $R_1(u)$  and  $R_2(u)$  near the point  $u = 0$  have the following form (we take  $\gamma = 0$ )

$$\check{R}_{u_0} = R_{1,2}(u, k)|_{u \rightarrow 0} P = I + u \begin{pmatrix} \pm \frac{\bar{\Delta}}{2\text{sn}[\mathbf{u}_0, \mathbf{k}]} & 0 & 0 & d'(0) \\ 0 & 0 & \frac{1}{\text{sn}[\mathbf{u}_0, \mathbf{k}]} & 0 \\ 0 & \frac{1}{\text{sn}[\mathbf{u}_0, \mathbf{k}]} & 0 & 0 \\ d'(0) & 0 & 0 & \mp \frac{\bar{\Delta}}{2\text{sn}[\mathbf{u}_0, \mathbf{k}]} \end{pmatrix}. \quad (4.11)$$

In the operator form the expansion of the matrix  $\check{R}_{u_0}$  is

$$\mathbf{R}_{u_0} = 1 \otimes 1 + u/2 (\mathbf{J}_1\sigma_1 \otimes \sigma_1 + \mathbf{J}_2\sigma_2 \otimes \sigma_2 + \mathbf{h}(\sigma_z \otimes 1 + 1 \otimes \sigma_z)). \quad (4.12)$$

Here  $J_1$  and  $J_2$  are characterized by parameter  $u_0$ ,

$$\mathbf{J}_1 = \frac{d'(0) \operatorname{sn}[u_0, k] + 1}{\operatorname{sn}[u_0, k]}, \quad \mathbf{J}_2 = \frac{1 - d'(0) \operatorname{sn}[u_0, k]}{\operatorname{sn}[u_0, k]}, \quad \mathbf{h} = \pm \frac{\bar{\Delta}}{2\operatorname{sn}[u_0, k]}.$$

The corresponding 1d quantum spin chain Hamiltonian has the form

$$H_{XY} = \sum_i (\mathbf{J}_1 \sigma_1[i] \sigma_1[i+1] + \mathbf{J}_2 \sigma_2[i] \sigma_2[i+1] + \mathbf{h} \sigma_3[i]). \quad (4.13)$$

In the physical aspects two cases of  $R_1$  and  $R_2$  are the same, and give  $XY$  model in the transverse field.

The  $R$ -matrices (3.12) with different  $u$  and  $v$  for the case  $b_0 = -1$  have the following algebraic formulation

$$\begin{aligned} \tilde{\mathbf{R}}(u; v) = & \frac{\cosh[u]}{2} (1 \otimes 1 + \sigma_3 \otimes \sigma_3) + i \frac{\sinh[u]}{2} (\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1) + \\ & \frac{\cos[v]}{2} (1 \otimes 1 - \sigma_3 \otimes \sigma_3) + \sin[v] (e^{\gamma/2} \sigma^+ \otimes \sigma^+ + e^{-\gamma/2} \sigma^- \otimes \sigma^-). \end{aligned} \quad (4.14)$$

At  $u = \pm iv$  and  $\gamma = \pm i\pi$  we arrive at  $\tilde{\mathbf{R}}(iv, v) = \cos[v] 1 \otimes 1 \pm \sin[v] \sigma_1 \otimes \sigma_2$  or  $\tilde{\mathbf{R}}(iv, v) = \cos[v] 1 \otimes 1 + \pm \sin[v] \sigma_2 \otimes \sigma_1$ , which coincide with the  $R$ -matrix, derived from the propagator in [18].

Setting  $u = -i\Theta v$  and  $\gamma = -2i\eta$ , and using the expansion of the operator (4.15) at the point  $v = 0$ , we shall come to the following spin-chain Hamiltonian operator

$$\begin{aligned} \tilde{H} = 1/2 \sum_i \Big( & \cos \eta (\sigma_1[i] \sigma_1[i+1] - \sigma_2[i] \sigma_2[i+1]) + \sin \eta (\sigma_1[i] \sigma_2[i+1] + \sigma_2[i] \sigma_1[i+1]) + \\ & \Theta (\sigma_1[i] \sigma_2[i+1] - \sigma_2[i] \sigma_1[i+1]) \Big). \end{aligned} \quad (4.15)$$

## 5 Exceptional cases

In the course of the previous analysis we have excluded some cases, which imply the following conditions on the matrix elements of the  $R$ -matrix:

$$h_i(u) = f_i(u) \quad \text{or} \quad h_i(u) = 0, \quad h, f = a, b, c, d, \quad i = 1, 2.$$

### 5.1 $h(u) = f(u), \quad h, f = a, b, c, d.$

In case of homogeneous matrix, when  $h_1(u) = h_2(u)$ ,  $h = a, b, c, d$ , one can find out some special spectral parameter-dependent solutions of YBE, which are not included in the parameterizations of the previous sections. We try to consider all possible solutions for the each discussed case. Among them, sometimes, there can be solutions, which are the limiting cases of the solutions discussed in the previous sections.

- If one take  $a(u) = c(u)$ , then from YBE the equations  $b(u) = \pm d(u)$  and

$$\frac{b(u)}{a(u)} - \frac{b(u+w)}{a(u+w)} + \frac{b(w)}{a(w)} \left( 1 - \frac{b(u)}{a(u)} \frac{b(u+w)}{a(u+w)} \right) = 0 \quad (5.16)$$

will follow. The solution is unique:  $\frac{b(u)}{a(u)} = \tanh u$ .

The resulting matrix is (up to a normalization function)

$$r_{a=c}(u) = \begin{pmatrix} 1 & 0 & 0 & \pm \tanh u \\ 0 & \tanh u & 1 & 0 \\ 0 & 1 & \tanh u & 0 \\ \pm \tanh u & 0 & 0 & 1 \end{pmatrix}. \quad (5.17)$$

This  $R$ -matrix has the free-fermionic property and is the particular case of the solutions (3.19, 3.23) with the parameters  $u_0 = K(k)$ ,  $k = 1$ .

The choice  $a(u) = -c(u)$  brings to constant solutions.

- Let  $a(u) = \pm b(u)$ . From YBE the relation  $d(u) = \pm c(u)$  follows, which is enough for the matrix to satisfy the all equations. So here, besides of the normalization function the solution contains one arbitrary function  $f(u)(= d(u)/c(u))$ :

$$r_{a=\pm b}(u) = \begin{pmatrix} 1 & 0 & 0 & \pm f(u) \\ 0 & \pm 1 & f(u) & 0 \\ 0 & f(u) & \pm 1 & 0 \\ \pm f(u) & 0 & 0 & 1 \end{pmatrix}. \quad (5.18)$$

In this case the matrix has free-fermionic property only for the constant function  $f(u) = \pm 1$ .

- The choice  $a(u) = \pm d(u)$  gives only constant solutions:  $c(u) = \pm a(u)$ ,  $b(u) = \pm a(u)$

up to the normalization function.

$$r_{a=\pm d}(u) = a(u) \begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & \pm 1 & \pm 1 & 0 \\ 0 & \pm 1 & \pm 1 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}. \quad (5.19)$$

- The other cases bring either to constant or already obtained solutions.

## 5.2 $h_i(u) = 0$

Here we permit (at least) the vanishing of one of the functions  $h_i(u)$ ,  $h = a, b, c, d$ . We are coming to the rather trivial solutions, when  $h = a, c$ .

**Constant solutions ( $u$  is included only in the phases).** Examples of the constant solutions are: when  $a_1(u) = 0$  (below  $\alpha, \gamma$  are arbitrary constants),

$$r_{a_1=0} = \begin{pmatrix} 0 & 0 & 0 & e^\gamma \\ 0 & \pm\sqrt{2} & \pm 1 & 0 \\ 0 & \pm 1 & \pm\sqrt{2} & 0 \\ e^{-\gamma} & 0 & 0 & 2 \end{pmatrix}, \quad (5.20)$$

or the matrix, when  $d_1 = 0$

$$r_{d_1=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \pm 1 & 0 \\ e^\gamma & 0 & 0 & 0 \end{pmatrix}. \quad (5.21)$$

A constant solution, when  $b_1(u) = b_2(0) = 0$  is

$$r_{b_1=0} = \begin{pmatrix} 1 & 0 & 0 & e^{\gamma \pm i\pi/4}/\sqrt{2} \\ 0 & 0 & e^{\pm i\pi/4}/\sqrt{2} & 0 \\ 0 & e^{\pm i\pi/4}/\sqrt{2} & 0 & 0 \\ e^{-\gamma \pm i\pi/4}/\sqrt{2} & 0 & 0 & \pm i \end{pmatrix}. \quad (5.22)$$



Some constant solutions with  $c_1(u) = 0$  (which have not too much zero matrix elements) are

$$r_{c_i=0} = \begin{pmatrix} 1 & 0 & 0 & e^{\gamma+\alpha u} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix} \Big|_{a_0^2=1}. \quad (5.23)$$

and

$$r_{c_1=0} = \begin{pmatrix} 1 & 0 & 0 & e^\gamma \\ 0 & a_0 & 0 & 0 \\ 0 & 1 + a_0 a_b & -a_b & 0 \\ 0 & 0 & 0 & a_0 a_b \end{pmatrix} \Big|_{a_0^2=1, a_b^2=1}. \quad (5.24)$$

Of course, one must remember that, all the matrices  $R$  are defined up to a multiplicative arbitrary function  $f(u)$ :  $R(u) \rightarrow f(u)R(u)$ , if  $R(u)$  is a solution of YBE, then the matrix  $f(u)R(u)$  also satisfies the equations.

For the constant solutions of YBE one can see the papers [19, 20].

### Spectral parameter dependent solutions

We below give all the such solutions, which have spectral parameter dependence (not only by an overall factor function or a phase function), provided that  $f_i(u) = 0$ , with some  $f = a, b, c, d$ .

**Homogeneous matrices.** A nontrivial solution arises when  $b_1(u) = b_2(0) = 0$ , which is rather similar to the XXZ model's  $R$ -matrix

$$r_{X-XZ}(u) = \begin{pmatrix} \sin[u_0] & 0 & 0 & \sin[u] \\ 0 & 0 & \sin[u+u_0] & 0 \\ 0 & \sin[u+u_0] & 0 & 0 \\ \sin[u] & 0 & 0 & \sin[u_0] \end{pmatrix}. \quad (5.25)$$

At the point  $u = 0$  the check version of the matrix (5.25) can be represented in following operator form,

$$\mathbf{r}_{\mathbf{X}-\mathbf{XZ}} = \sin[u_0]1 \otimes 1 + u(\mathbf{J}(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) - \Delta\sigma_3 \otimes \sigma_3). \quad (5.26)$$

Here  $\mathbf{J} = 1$  and  $\Delta = \cos[u_0]$ . So, this  $R$ -matrix describes a  $XYZ$  model with  $J_1 = -J_2$ .

$$H_{X-XZ} = \sum_i (\mathbf{J}(\sigma_1[i] \otimes \sigma_1[i] - \sigma_2[i] \otimes \sigma_2[i]) - \Delta \sigma_z[i] \otimes \sigma_z[i]), \quad (5.27)$$

One can see, that  $\bar{R} = r_{X-XZ}(u)$  and the  $R$ -matrix of the  $XXZ$  model can be connected one with other by this transformation  $\bar{R}_{ij}^{i'j'} = R_{i\bar{j}}^{i'\bar{j}'}$ , where  $\bar{i} = \text{mod}[i+1]2$ , i.e. it interchanges the indexes 0 and 1.

Note that the solution (5.25) overlaps with the inhomogeneous solution (3.12). At  $u_0 = \pi/2$  the matrix (5.25) coincides with the  $\tilde{R}(0, u)$ . An arbitrary constant  $e^\gamma$  can be included also in  $r_{X-XZ}(u)$ :  $d_1 \rightarrow e^\gamma d_1$ ,  $d_2 \rightarrow e^{-\gamma} d_2$ .

**Inhomogeneous matrices.** A rational solution is obtained from the condition  $b_1 = 0$ , if it is assumed, that  $b_2(u) \neq 0$  and instead of it the relations  $d_1(u) = d_2(u) = 0$  take place.

$$r_{b_1=d_i=0}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.28)$$

A simple solution one can find out when,  $b_i = 0$  and  $d_i = 0$ ,  $i = 1, 2$ ,

$$r_{b_i=d_i=0} = \begin{pmatrix} e^{p \cdot u} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{q \cdot u} \end{pmatrix}, \quad (5.29)$$

where  $p$  and  $q$  are arbitrary numbers. A particular case with  $q = \pm p$  of this matrix can be obtained from the solution  $R_{XXZ}$  (2.14), dividing it onto  $\sin[u_0]$  and taking the limit  $u_0 \rightarrow \infty$ .

The case  $b_i = 0$ ,  $i = 1, 2$  and  $d_i \neq 0$ , when  $a_1 \neq a_2$  leads to elliptic solutions. Here the number of the independent YB equations is four (we set  $c_i = 1$ ,  $d_1 = d_2 \equiv d$ ):

$$a_1(u+w) - a_1(u)a_1(w) - a_2(u+w)d(u)d(w) = 0, \quad (5.30)$$

$$\begin{aligned}
a_2(u+w) - a_2(u)a_2(w) + a_1(u+w)d(u)d(w) &= 0, \\
a_1(u)d(w) - a_2(u)d(u+w) + a_2(u+w)a_2(w)d(u) &= 0, \\
a_2(u)d(w) - a_1(u)d(u+w) + a_1(u+w)a_1(w)d(u) &= 0.
\end{aligned} \tag{5.31}$$

Finding that  $a_i(0) = 1$  and  $d(0) = 0$ , and expanding the equations near the point  $u = 0$ , it follows that the function  $d(u)$  must satisfy the following differential equation  $d''(u) = 2(\bar{a}_0)^2 d(u)^3 + 2((\bar{a}_0)^2 + 2a_0^2)d(u)$ , with some definite  $a_0$  and  $\bar{a}_0$ . The solution of this equation is expressed by the Jacobi's elliptic function:  $d(u) = \bar{a}_0 \text{sn}[u, k]$ . Then the functions  $a_1(u)$  and  $a_2(u)$  are given by the following relations

$$a_1(2u) = \frac{2a_0 d(u)}{\bar{a}_0(1+d(u)^2)} + \frac{d'(u)}{\bar{a}_0(1-d(u)^2)}, \quad a_2(2u) = \frac{-2a_0 d(u)}{\bar{a}_0(1+d(u)^2)} + \frac{d'(u)}{\bar{a}_0(1-d(u)^2)}.$$

The values of the parameters  $a_0$  and  $\bar{a}_0$  are expressed by the module  $k$  as

$$\bar{a}_0 = \pm i\sqrt{k} \quad \& \quad a_0 = \pm i(k-1)/2 \quad \text{and} \quad \bar{a}_0 = \pm\sqrt{k} \quad \& \quad a_0 = \pm i(k+1)/2.$$

These solutions can be obtained from the matrix (3.23) taking  $u_0 = iK'$ ,  $\text{sn}[iK', k] = \infty$  and making the Landen's transformation of the elliptic functions,  $k \rightarrow 2\sqrt{k}/(1+k)$ ,  $u \rightarrow (1+k)u$  [4].

$d_i(u) = 0$ . Trigonometric solutions, similar to the  $XXZ$  model's  $R$ -matrix, exist when the constraints  $d_2(u) = 0$ ,  $d_1 \neq 0$  take place. From the equations in (2.1), which contain only the functions  $c_1(u)$ ,  $c_2(u)$ , one finds out  $c_2(u) = c_1(u)e^{\alpha u}$ . After comparison of some equations in (2.32), one comes to  $b_2(u) = b_1(u)b_0$ ,  $b_0$  being a constant (if  $c_i(u) \neq 0$ ). Then, in the same way as previously, we are coming to the relation (2.9), i.e.  $\frac{a_2(u)-a_1(u)}{b(u)} = \bar{\Delta}$  is a constant. Now let us consider two cases with zero or non-zero values of  $\bar{\Delta}$  separately.

**The case  $\bar{\Delta} = 0$ .** Then the independent equations in the set (2.32) are the following ones (provided that  $d_1 \neq 0$ ).

$$a_1(u)b_1(u+w)c_1(w) - b_1(w)c_1(u)c_1(u+w) - a_1(u+w)b_1(u)c_1(w) = 0,$$

$$\begin{aligned}
a_1(u)a_1(w)c_1(u+w) - b_0b_1(w)b_1(u)c_1(u+w) - a_1(u+w)c_1(u)c_1(w) &= 0, \\
b_0a_1(w)b_1(u+w)c_1(u) - b_0a_1(u+w)b_1(w)c_1(u) - b_0b_1(u)c_1(u+w)c_1(w) &= 0, \\
a_1(w)c_1(u+w)d_1(u) - a_1(w)c_1(u)d_1(u+w) + a_1(u)a_1(u+w)d_1(w) - b_1(u)b_1(u+w)d_1(w) &= 0, \\
b_0b_1(u+w)c_1(w)d_1(u) + a_1(u)b_1(w)d_1(u+w) - b_0a_1(w)b_1(u)d_1(u+w) - b_1(u+w)c_1(u)d_1(w) &= 0, \\
a_1(u+w)a_1(w)d_1(u) - b_1(u+w)b_1(w)d_1(u) - a_1(u)c_1(w)d_1(u+w) + a_1(u)c_1(u+w)d_1(w) &= 0.
\end{aligned} \tag{5.32}$$

First three equations coincide with the homogeneous ones in (2.3). So the relation (2.6) takes place and the unique non-trivial solutions can be parameterized as the matrix elements of the  $XXZ$  model -  $a_1(u) = \sin[u + u_0]$ ,  $b_1(u) = \sin[u]$ ,  $c_1 = \sin[u_0]$ . The analysis of the remaining three equations gives the following solution for the next function,  $d_1(u) = \sin[u + u_0] \sin[u]e^\varepsilon$ , where  $\varepsilon$  is a constant number. The matrix representation of this solution is

$$r_{XXZ/d}(u) = \begin{pmatrix} \sin[u + u_0] & 0 & 0 & \sin[u + u_0] \sin[u]e^\varepsilon \\ 0 & \sin[u] & \sin[u_0] & 0 \\ 0 & \sin[u_0] & \sin[u] & 0 \\ 0 & 0 & 0 & \sin[u + u_0] \end{pmatrix}. \tag{5.33}$$

The operator representation of the expansion of the check matrix at the point  $u = 0$  looks like as follows

$$\mathbf{r}_{XXZ/d} = \sin[u_0]1 \otimes 1 + u((\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) - \Delta\sigma_3 \otimes \sigma_3) + e^\varepsilon\sigma_+ \otimes \sigma_+. \tag{5.34}$$

Here  $\Delta = \cos[u_0]$  and  $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ . The corresponding one dimensional spin-Hamiltonian is

$$H_{XXZ/d} = \sum_i ((\sigma_1[i]\sigma_1[i+1] + \sigma_2[i]\sigma_2[i+1]) - \Delta\sigma_z[i]\sigma_z[i+1] + e^\varepsilon\sigma_+[i]\sigma_+[i+1]). \tag{5.35}$$

With the general parametrization of the  $XXZ$ -model  $a_1(u) = a_2(u) = \sin[u + u_0]$ ,  $b_1(u) = \sqrt{b_0} \sin[u]$ ,  $b_2(u) = \sin[u]/\sqrt{b_0}$ ;  $c_1 = e^{\alpha u} \sin[u_0]$ ,  $c_2(u) = e^{-\alpha u} \sin[u_0]$ , one will find easily that the only solution to (5.32) corresponds to the choice  $\alpha = 0$ ,  $b_0 = 1$ , i.e. coincides with the solution given above.

But we can check, that there is also another solution for the particular case  $\cos [u_0] = 0$ . Then it turns out  $b_0 = -1$ . Corresponding  $R$ -matrix looks like ( $\alpha$  is an arbitrary number)

$$\tilde{r}_{XX/d}(u) = \begin{pmatrix} \cos [u] & 0 & 0 & \sinh [\alpha u] e^\varepsilon \\ 0 & \pm i \sin [u] & e^{\alpha u} & 0 \\ 0 & e^{-\alpha u} & \mp i \sin [u] & 0 \\ 0 & 0 & 0 & \cos [u] \end{pmatrix}. \quad (5.36)$$

**The case  $\bar{\Delta} \neq 0$ .** The set of the independent equations now are

$$\begin{aligned} a_1(u)b_1(u+w)c_1(w) - b_1(w)c_1(u)c_1(u+w) - a_1(u+w)b_1(u)c_1(w) &= 0, \\ a_1(u)a_1(w)c_1(u+w) - b_1(w)b_2(u)c_1(u+w) - a_1(u+w)c_1(u)c_1(w) &= 0, \\ a_1(w)b_2(u+w)c_1(u) - a_1(u+w)b_2(w)c_1(u) - b_2(u)c_1(u+w)c_1(w) &= 0, \\ a_2(u+w)b_1(w)c_1(u) + b_1(u)c_1(u+w)c_1(w) - a_2(w)b_1(u+w)c_1(u) &= 0, \\ b_1(u)b_2(w)c_1(u+w) + a_2(u+w)c_1(u)c_1(w) - a_2(u)a_2(w)c_1(u+w) &= 0, \\ a_2(u)b_2(u+w)c_1(w) - b_2(w)c_1(u)c_1(u+w) - a_2(u+w)b_2(u)c_1(w) &= 0, \\ a_2(w)c_1(u+w)d_1(u) - a_1(w)c_1(u)d_1(u+w) + a_1(u)a_1(u+w)d_1(w) - b_1(u)b_1(u+w)d_1(w) &= 0, \\ b_2(u+w)c_1(w)d_1(u) + a_1(u)b_1(w)d_1(u+w) - a_1(w)b_2(u)d_1(u+w) - b_1(u+w)c_1(u)d_1(w) &= 0, \\ b_2(u+w)b_2(w)d_1(u) + a_1(u)c_1(w)d_1(u+w) - a_2(u)c_1(u+w)d_1(w) - a_1(u+w)a_1(w)d_1(u) &= 0, \\ a_2(u+w)a_2(w)d_1(u) - b_1(u+w)b_1(w)d_1(u) - a_2(u)c_1(w)d_1(u+w) + a_1(u)c_1(u+w)d_1(w) &= 0, \\ a_2(w)b_1(u)d_1(u+w) - a_2(u)b_2(w)d_1(u+w) + b_2(u+w)c_1(u)d_1(w) - b_1(u+w)c_1(w)d_1(u) &= 0, \\ a_2(w)c_1(u)d_1(u+w) - a_2(u)a_2(u+w)d_1(w) + b_2(u)b_2(u+w)d_1(w) - a_1(w)c_1(u+w)d_1(u) &= 0. \end{aligned} \quad (5.37)$$

The solutions of the first six equations are known. Let us take most general parameterizations of them as  $a_1(u) = \sin [u + u_0]$ ,  $b_1(u) = \sqrt{b_0} \sin [u]$ ,  $b_2(u) = \sin [u]/\sqrt{b_0}$ ;  $c_1 = e^{\alpha u} \sin [u_0]$ ,  $c_2(u) = e^{-\alpha u} \sin [u_0]$ ,  $a_2(u) = \sin [u_0 - u]$ . Then from the next two equations we are finding a relation  $d(u) = e^{\alpha u} \csc [u_0] \sin [u] d'(0) (\sin [u + u_0] + b_0 \sin [u_0 - u]) / (1 + b_0)$ . The remaining equations are giving constraints on the values of  $\alpha$  and  $b_0$ . Finally we are

arriving at the following solutions. For  $\alpha = 0$  and  $b_0 = 1$ , the  $R$ -matrix is

$$r_{XYZ/d}(u) = \begin{pmatrix} \sin[u + u_0] & 0 & 0 & \cos[u] \sin[u] e^\varepsilon \\ 0 & \sin[u] & \sin[u_0] & 0 \\ 0 & \sin[u_0] & \sin[u] & 0 \\ 0 & 0 & 0 & \sin[u_0 - u] \end{pmatrix}, \quad (5.38)$$

and for the pairs  $\alpha = i$  and  $b_0 = e^{2iu_0}$  and  $\alpha = -i$ ,  $b_0 = e^{-2iu_0}$  we find

$$r_{XYZ/d}^\pm(u) = \begin{pmatrix} \sin[u + u_0] & 0 & 0 & \sin[u] e^\varepsilon \\ 0 & \sin[u] e^{\pm iu_0} & \sin[u_0] e^{\pm iu} & 0 \\ 0 & \sin[u_0] e^{\mp iu} & \sin[u] e^{\mp iu_0} & 0 \\ 0 & 0 & 0 & \sin[u_0 - u] \end{pmatrix}, \quad (5.39)$$

where  $\varepsilon$  is a constant, which is defined as  $e^\varepsilon = d''(0)$ .

Of course, the transpositions of the matrices (5.38) and (5.39) are also solutions of YBE.

The  $R$ -matrix in (5.38) can be found from the matrix (3.23) by taking an appropriate limit  $r_{XYZ/d}(u) = \lim_{k \rightarrow 0, e^\gamma = e^\varepsilon/k} R_2(u, k)$ . In contrast to this, the matrix (5.39) is not included in the limiting set of the solutions obtained in the previous sections. Note, that the solutions (5.39) and (5.36) overlaps at the particular values of the parameters  $u_0$  and  $\alpha$ ,  $u_0 = \pm\pi/2$  and  $\alpha = \pm i$ .

The operator representation of the check matrix of (5.38) at the point  $u = 0$  looks like

$$\mathbf{r}_{\mathbf{XYZ}/d} = \sin[u_0] 1 \otimes 1 + u (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \Delta/2 (\sigma_3 \otimes 1 + 1 \otimes \sigma_3) + e^\varepsilon \sigma_+ \otimes \sigma_+) \quad (5.40)$$

Here, as in the previous case,  $\Delta = \cos[u_0]$  and  $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ . The corresponding one dimensional spin-Hamiltonian is

$$H_{XYZ/d} = \sum_i (\sigma_1[i] \sigma_1[i+1] + \sigma_2[i] \sigma_2[i+1] + \Delta \sigma_3[i] + e^\varepsilon \sigma_+[i] \sigma_+[i+1]). \quad (5.41)$$

The one-dimensional spin-Hamiltonian appropriate for the  $R$ -matrix (5.39) contains some additional terms

$$H_{XYZ/d}^\pm = \sum_i (\sigma_1[i] \sigma_1[i+1] + \sigma_2[i] \sigma_2[i+1] \pm i/2 (\sigma_3[i] - \sigma_3[i+1])) \quad (5.42)$$

$$+(e^{\pm iu_0} - 1)\sigma_+[i]\sigma_-[i+1] + (e^{\mp iu_0} - 1)\sigma_-[i]\sigma_+[i+1] + \Delta\sigma_3[i] + e^\varepsilon\sigma_+[i]\sigma_+[i+1]).$$

## 6 Tetrahedral algebra of Zamolodchikov

There are very few solutions to the Zamolodchikovs' tetrahedral equations (ZTE) [22, 23]. Their vertex versions can be served as the analogs of the vertex Yang-Baxter equations for three dimensional case. The solutions to ZTE, as well as the solutions to the equations presented in [24] ensure the integrability of the models constructed therein. One of established ways of looking for the solutions of ZTE is connected with the use of solutions of YBE [23, 25]. The brief description of it is the following. If  $R_{k\ p}^i(u, v)$ ,  $i = 1, 2$ , are matrices acting non-trivially on the space  $V_k \otimes V_k$ , and have following properties

$$R_{12}^1(u, v)R_{13}^1(u, w)R_{23}^1(v, w) = R_{23}^1(v, w)R_{13}^1(u, w)R_{12}^1(u, v), \quad (6.1)$$

$$R_{12}^1(u, v)R_{13}^2(u, w)R_{23}^2(v, w) = R_{23}^2(v, w)R_{13}^2(u, w)R_{12}^1(u, v), \quad (6.2)$$

then the matrix  $W_{i_1 j_1 k_1}^{i_2 j_2 k_2}$ , obtained from the relation, known as tetrahedral Zamolodchikov's algebra

$$R_{12}^{i_1}(u, v)R_{13}^{j_1}(u, w)R_{23}^{k_1}(v, w) = \sum_{i_2 j_2 k_2} W(u, v, w)_{i_1 j_1 k_1}^{i_2 j_2 k_2}(u, v, w)R_{23}^{k_2}(v, w)R_{13}^{j_2}(u, w)R_{12}^{i_2}(u, v) \quad (6.3)$$

can be served as a good candidate to the solution of the vertex version of ZTE

$$\begin{aligned} & W_{i_1 i_2 i_3}^{j_1 j_2 j_3}(u_1, u_2, u_3)W_{j_1 i_4 i_5}^{k_1 j_4 j_5}(u_1, u_2, u_4)W_{j_2 j_4 i_6}^{k_2 k_4 j_6}(u_1, u_3, u_4)W_{j_3 j_5 j_6}^{k_3 k_5 k_6}(u_2, u_3, u_4) \\ &= W_{i_3 i_5 i_6}^{j_3 j_5 j_6}(u_2, u_3, u_4)W_{i_2 i_4 j_6}^{j_2 j_4 k_6}(u_1, u_3, u_4)W_{i_1 j_4 j_5}^{j_1 k_4 k_5}(u_1, u_2, u_4)W_{j_1 j_2 j_3}^{k_1 k_2 k_3}(u_1, u_2, u_3). \end{aligned} \quad (6.4)$$

At least this relation will be hold on the space

$$R_{34}^{k_1}(u_3, u_4)R_{24}^{k_2}(u_2, u_4)R_{14}^{k_3}(u_1, u_4)R_{2,3}^{k_4}(u_2, u_3)R_{1,3}^{k_5}(u_1, u_3)R_{12}^{k_6}(u_1, u_2) |v_1\rangle|v_2\rangle|v_3\rangle|v_4\rangle,$$

which can be easily verified repeatedly using the equality (6.3). The written product of the  $R_{ij}$ -matrices is defined on the tensor product of two-dimensional spaces  $(V_1 \otimes V_2 \otimes V_3 \otimes V_4)$ .

For the matrix  $R^1(u, v)$  coinciding with the homogeneous  $XY$ -model's  $R_{XY}(u - v)$ , the corresponding  $R^2(u, v)$  and  $W(u, v, w)$ -matrices are found [23, 25] and it is proven, that  $W(u, v, w)$  is a solution of ZTE with  $R^2(u, v)$  chosen as

$$R^2(u, v)_{i_1 j_1}^{i_2 j_2} = (-1)^{j_1+1} R^1(u + v)_{i_1 j_1}^{i_2 j_2}. \quad (6.5)$$

In the same time the following equations take place

$$R_{12}^2(u, v) R_{13}^1(u, w) R_{23}^1(v, w) = R_{23}^1(v, w) R_{13}^1(u, w) R_{12}^{2\tau}(u, v), \quad (6.6)$$

where

$$R^{2\tau}(u, v)_{i_1 j_1}^{i_2 j_2} = (-1)^{i_1+1} R^1(u + v)_{i_1 j_1}^{i_2 j_2}.$$

Choosing the matrix  $R_{ij}^1$  as one of the inhomogeneous  $R_{ij}$ -matrices, discussed in this paper, and taking it's pair  $R_{ij}^2$  in the same way as in (6.5), we come to the conclusion, that the  $W(u, v, w)$ -matrices satisfying to (6.3) exist for the particular cases which coincide or are equivalent to the homogeneous cases [23, 25] after some automorphisms (mainly it brings to the change of the signs of some matrix elements of  $W(u, v, w)$ ). In case of the general inhomogeneous matrices, there are no such  $W(u, v, w)$ -matrices.

Here we present another way of choosing of the pair  $R^i$ , for which there is a  $W(u, v, w)$ -matrix satisfying Eq.(6.3). However this matrix is not a solution of ZTE, but it is an example, that it is possible to construct the algebra (6.3) by choosing  $R_{ij}^i(u, v)$ -matrices in various ways. By direct calculations it can be verified, that the matrices

$$R^1(u_1, u_2) = \begin{pmatrix} \cos[u_1 - u_2] & 0 & 0 & 0 \\ 0 & i \sin[u_1 - u_2] & 1 & 0 \\ 0 & 1 & -i \sin[u_1 - u_2] & 0 \\ 0 & 0 & 0 & \cos[u_1 - u_2] \end{pmatrix}, \quad (6.7)$$

$$R^2(u_1, u_2) = \begin{pmatrix} 1 & 0 & 0 & \sin[u_1 - u_2] \\ 0 & 0 & \cos[u_1 - u_2] & 0 \\ 0 & \cos[u_1 - u_2] & 0 & 0 \\ \sin[u_1 - u_2] & 0 & 0 & 1 \end{pmatrix}, \quad (6.8)$$



which are particular cases of the matrix (3.12), both are the solutions of ordinary YBE, but for them the relation (6.2) does not take place. The following (unique)  $W(u, v, w)$ -matrix ensures the validity of the tetrahedral Zamolodchikov algebra.

$$\begin{aligned}
W_{000}^{000} &= W_{111}^{111} = 1, \\
W_{ijk}^{i'j'k'} &= \bar{W}_{ijk}^{i'j'k'} (2f[u_1, u_2, u_3] - \sin[2(u_1 - u_2)] \sin[2(u_2 - u_3)])^{-1}, \\
f[u_1, u_2, u_3] &= \sin[u_1 - u_2]^2 + \sin[u_2 - u_3]^2 + \sin[u_1 - u_3]^2, \\
\bar{W}_{001}^{001} &= 2f[u_1, u_2, u_3] \cos[u_2 - u_3] \csc[u_1 - u_3] \sin[u_1 - u_2], \\
\bar{W}_{010}^{010} &= -\sin[2(u_1 - u_2)] \sin[2(u_2 - u_3)], \quad \bar{W}_{001}^{010} = \cos[u_1 - u_2] \sin[u_2 - u_3] \sin[u_1 - u_3], \\
\bar{W}_{010}^{001} &= 2f[u_1, u_2, u_3] \csc[u_1 - u_3]^2 \sin[u_2 - u_3]^2, \quad \bar{W}_{000}^{011} = -4 \sin[u_1 - u_2]^2 \sin[u_2 - u_3]^2, \\
\bar{W}_{011}^{011} &= 2f[u_1, u_2, u_3] \cos[u_1 - u_2] \csc[u_1 - u_3] \sin[u_2 - u_3], \\
\bar{W}_{100}^{001} &= -4 \cos[u_2 - u_3] \csc[u_1 - u_3] \sin[u_1 - u_2] \sin[u_2 - u_3]^2, \\
\bar{W}_{100}^{010} &= 4 \cos[u_2 - u_3] \sin[u_1 - u_2] \sin[u_1 - u_3], \quad \bar{W}_{111}^{001} = -4 \csc[u_1 - u_3]^2 \sin[u_2 - u_3]^2, \\
\bar{W}_{111}^{010} &= 4, \quad \bar{W}_{101}^{011} = 2f[u_1, u_2, u_3], \quad \bar{W}_{110}^{011} = -4 \cos[u_1 - u_2] \csc[u_1 - u_3] \sin[u_2 - u_3]^2, \\
\bar{W}_{001}^{100} &= -4 \cos[u_1 - u_2] \csc[u_1 - u_3] \sin[u_1 - u_2]^2 \sin[u_2 - u_3], \\
\bar{W}_{000}^{101} &= 4 \sin[u_1 - u_2] \sin[u_2 - u_3]^2, \quad \bar{W}_{010}^{100} = 2f[u_1, u_2, u_3] \csc[u_1 - u_3]^2 \sin[u_1 - u_2]^2, \\
\bar{W}_{011}^{101} &= 4 \cos[u_2 - u_3] \csc[u_1 - u_3] \sin[u_1 - u_2]^3, \quad \bar{W}_{000}^{110} = -4 \sin[u_1 - u_2]^2 \sin[u_2 - u_3]^2, \\
\bar{W}_{011}^{110} &= -4 \cos[u_2 - u_3] \csc[u_1 - u_3] \sin[u_1 - u_2]^3, \quad \bar{W}_{111}^{100} = -4 \csc[u_1 - u_3]^2 \sin[u_1 - u_2]^2, \\
\bar{W}_{100}^{100} &= -2f[u_1, u_2, u_3] \cos[u_1 - u_2] \csc[u_1 - u_3] \sin[u_2 - u_3], \\
\bar{W}_{101}^{101} &= -\sin[2(u_1 - u_2)] \sin[2(u_2 - u_3)], \quad \bar{W}_{110}^{101} = 4 \cos[u_1 - u_2] \csc[u_1 - u_3] \sin[u_2 - u_3], \\
\bar{W}_{101}^{110} &= 2f[u_1, u_2, u_3], \quad \bar{W}_{110}^{110} = 2f[u_1, u_2, u_3] \cos[u_2 - u_3] \csc[u_1 - u_3] \sin[u_1 - u_2].
\end{aligned} \tag{6.10}$$

## 7 Conclusions

In this article we give all possible solutions of the YBE with general inhomogeneous spectral parameter dependent  $R(u)$ -matrix corresponding to the six-and eight-vertex models. The symmetry relations imposed by the Yang-Baxter equations on the elements of the general inhomogeneous  $R$ -matrices, together with the consistency conditions are obtained. Thus, we present more complete classification of the solutions, than it was done before. The main conclusion about the nature of the solutions is, that besides of the known homogeneous solutions (corresponding to the  $XXZ$  and  $XYZ$  models), which admit redefinitions (including parameters  $\alpha$ ,  $b_0$ ,  $d_0$ ) due to gauge transformations, the all other solutions - inhomogeneous or homogeneous, with the behavior  $\check{R}(0) \approx I$  (important in the context of the integrable theory), have the "free-fermionic" property. As it is known, this property ensures, that the corresponding physical models are exactly solvable [13, 12]. The fact that the "free-fermionic"  $R$ -matrices satisfy to the Yang-Baxter equations, hints that they admit some underlying symmetry [14, 21]. It is remarkable, that among the exceptional solutions discussed in the Section 5 we met such spectral-parameter dependent solutions (5.18, 5.36, 5.39) which are not limit cases of the solutions with general structure. The one-parametric solution (3.11), obtained in the subsection 3.2 gives rise to the two-parametric solution (3.12) to the YBE (3.13). In the Section 6 we discuss the possibility to use the obtained matrices for the construction of the solutions to the Zamolodchikov's Tetrahedral Algebra. All the quantum chain-models constructed with the obtained inhomogeneous  $R(u)$ -matrices ( $R_{00}^{00} \neq R_{11}^{11}$ ) describe spin-1/2 models with nearest-neighbor interactions in a transverse magnetic field. By means of the spin-fermion transformations, the corresponding Hamiltonian operators describe nearest-neighbor interactions of free spin-less fermions on a chain.

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